Task-Space Inverse Dynamics

Optimization-based Robot Control

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From Joint Space to Task Space Control

Limits of Joint-Space Control

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What if we have reference trajectory $x^r(t)$ for end-effector?

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Tracking $q^r(t)$ is sufficient but not necessary to track $x^r(t)$: controller rejects also perturbations affecting q without affecting $FG(q)$.

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Finally compute joint torques as:

$$\tau = M\dot{v}^d + h \tag{5}$$

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- More complex controller

End-Effector Control as LSP

End-effector control law (Option 2):

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can be computed as:

$$\begin{aligned} \underset{\tau,\dot{v}}{\text{minimize}} \quad & ||J\dot{v} + \dot{J}v - \dot{\nu}^d||^2 \\ \text{subject to} \quad & M\dot{v} + h = \tau \end{aligned} \tag{10}$$

Task Models

Task-Function Approach

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N.B.

Here: e depends on instantaneous state-control value.
In optimal control: e depends on state-control trajectory.

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Three kinds of task functions:

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Solution

Impose dynamics of $e(x, t)$ (e.g., $\dot{e} = ...$)
which should be affine function of $\dot{v}$
such that $\lim_{t \to \infty} e(x, t) = 0$

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End up with affine function of $\dot{v}$ and u:

$$g(y) \triangleq \underbrace{\begin{bmatrix} A_v & A_u \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \dot{v} \\ u \end{bmatrix}}_{y} - a$$

Optimization-Based Control

Task-Space Inverse Dynamics (TSID)

Find τ that minimizes task function:

$$\begin{aligned} \underset{y=(\dot{v},\tau)}{\text{minimize}} \quad & ||Ay - a||^2 \\ \text{subject to} \quad & \begin{bmatrix} M & -S^\top \end{bmatrix} y = -h \end{aligned} \tag{13}$$

TSID for Robots in Soft Contact

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If contacts are soft, use estimated forces $\hat{f}$:

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Introduce forces and constraints:

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Inequality Constraints

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Any inequality affine in $y = (\tau, f, \dot{v})$:

- joint torque bounds: $\tau^{min} \leq \tau \leq \tau^{max}$
- (linearized) force friction cones: $Bf \leq 0$
- joint bounds: $\dot{v}^{min} \leq \dot{v} \leq \dot{v}^{max}$
- collision avoidance (more complicated)

Multi-Task Control

Multi-Objective Optimization

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Can use redundancy to execute secondary tasks, but how?

Weighted Multi-Objective Optimization

N tasks, each defined by task function

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CONS Hard to find weights $\rightarrow$ too large/small weights lead to numerical issues.

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Computational Aspects

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- $n_v + n_{va} + n_f$ variables (≈ 70 for humanoid)
- $n_v + n_f$ equality constraints (≈ 40 for humanoid)
- $n_v + n_{va} + \frac{4}{3} n_f$ inequality constraints (*assuming friction cones approximated with 4-sided pyramids*)

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QUESTIONS

- Can we solve itin 1 ms?
- Can we speed up computation?

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IDEA: Exploit problem structure to speed up computation.

Equality constraints have special structure:

$$\begin{bmatrix} J & 0 & 0 \\ M_u & -J_u^\top & -0 \\ M_a & -J_a^\top & -I \end{bmatrix} \begin{bmatrix} \dot{v} \\ f \\ \tau \end{bmatrix} = \begin{bmatrix} -\dot{J}v \\ -h_u \\ -h_a \end{bmatrix}$$

Identity matrix is easy to invert → Easy to express τ as affine function of other variables.

$$\underbrace{\begin{bmatrix} \dot{v} \\ f \\ \tau \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} I & 0 \\ 0 & I \\ M_a & -J_a^\top \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \dot{v} \\ f \end{bmatrix}}_{\bar{y}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ h_a \end{bmatrix}}_{d}$$

Reformulating Optimization Problem

Original problem:

$$
\begin{aligned}
\underset{y}{\text{minimize}} \quad & ||Ay - a||^2 \\
\text{subject to} \quad & By \leq b \\
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Use $y = D\bar{y} + d$ to reformulate as [2]:

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Removed n_{va} variables and n_{va} equality constraints!

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Yes:

- for floating-base, remove first 6 variables of $\dot{v}$ exploiting structure of first 6 columns of M_u
- remove (either all [3, 4] or some [1]) force variables by projecting dynamics in null space of J

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BUT these tricks either limit expressiveness, or lead to small improvements (while making software more complex).

My opinion: not worth it!

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SOLUTION Represent SE(3) elements using homogeneous matrices $y \in \mathbb{R}^{4\times 4}$ and redefine error function:

$$e(q, t) = \log(y^*(t)^{-1} y(q)),$$

where $\log \triangleq$ inverse operation of matrix exponential (i.e. exponential map): transforms displacement into twist.

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