Robot Modeling

Optimization-based Control of Legged Robots

Andrea Del Prete

University of Trento, 2023

Schedule

Classroom Code: 4qb7dbt

First part:

1. Modeling ($\approx$ 1 hour)
2. Joint-Space Control ($\approx$ 1 hour)
3. Task-Space Control ($\approx$ 1 hour)
4. Implementation ($\approx$ 1 hour)
5. Coding ($\approx$ 2 hours)

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Second part:

1. Limits of Reactive Control ($\approx$ 0.5 hour)
2. Linear Inverted Pendulum Model $\approx$ 0.5 hour)
3. Center of Mass Trajectory Generation ($\approx$ 1 hour)
4. Implementation ($\approx$ 1 hour)
5. Coding: CoM trajectory optimization ($\approx$ 1 hour)
6. Coding: walking with TSID ($\approx$ 2 hours)

Options for coding

- use provided docker image
  - `docker pull andreadelprete/orc23:casadi`
  - password: iamarobot
  - Follow instructions on shared folder

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- install TSID and dependencies on your Ubuntu machine
  - either with Debian packages: `sudo apt install ...`
  - or compiling source code available on *github.com*:
    - TSID
    - Pinocchio
    - Gepetto-viewer
    - Gepetto-viewer-corba
    - example-robot-data

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Notation & Definitions

State $\triangleq x$.

Control $\triangleq u$.

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Zero matrix $\triangleq 0$.
Matrix size written as index (when needed), e.g., I_3.

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Fully actuated system: number of actuators $=$ number of degrees of freedom (e.g., manipulator).

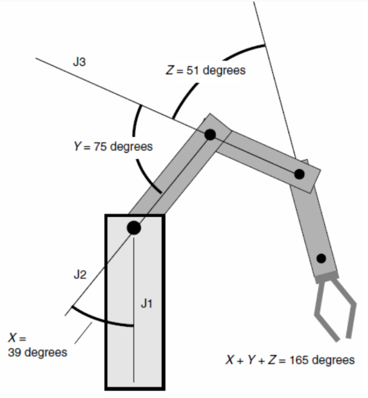
Under actuated system: number of actuators $<$ number of degrees of freedom (e.g., legged robot, quadrotor).

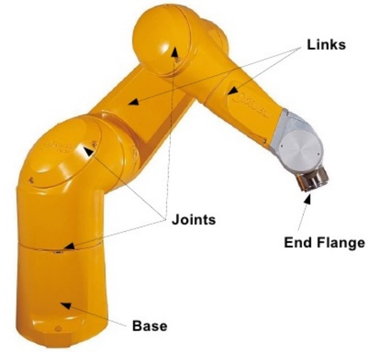
Modeling Robot Manipulators

Robot Manipulators: Fixed-base Robots

Robot base is (typically) fixed (e.g., attached to the ground).

Configuration represented by vector $q \in \mathbb{R}^{n_q}$ of (relative) joint angles.
Velocity represented by vector $v = \dot{q} \in \mathbb{R}^{n_v}$ of (relative) joint velocities.





Actuation Models

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Appropriate model depends on robot and task.

Velocity Input

Model actuators as velocity sources.

- Good for hydraulic.
- Good for electric in certain conditions (e.g., manipulators).

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$$x \triangleq q$$

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Dynamics is simple integrator:

$$\dot{x} = u$$

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Model actuators as acceleration sources.

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$$x \triangleq (q, v)$$

$$u \triangleq \dot{v}$$

Dynamics is double integrator:

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

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Good for electric w/o high-friction gear box—rarely the case (unfortunately).

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Dynamics of fully-actuated mechanical system (e.g., manipulator):

$$M(q)\dot{v} + h(q, v) = \tau,$$

where

- $M(q) \in \mathbb{R}^{n_v \times n_v} \triangleq$ (positive-definite) mass matrix,
- $h(q, v) \in \mathbb{R}^{n_v} \triangleq$ bias forces,
- $\tau \in \mathbb{R}^{n_v} \triangleq$ joint torques.

Torque Input: Fully-Actuated Dynamics

Bias forces sometimes decomposed as:

$$h(q, v) = C(q, v)v + g(q)$$

- $C(q, v)v \triangleq$ Coriolis and centrifugal effects
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Nonlinear state-space dynamics:

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -M(q)^{-1}h(q, v) \end{bmatrix} + \begin{bmatrix} 0 \\ M(q)^{-1} \end{bmatrix} u$$

Inverse VS Forward Dynamics

Forward Dynamics
Given q, v, τ compute $\dot{v}$:

$$\dot{v} = M(q)^{-1}(\tau - h(q, v))$$

Problem solved by simulators.

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Inverse Dynamics
Given $q, v, \dot{v}$ compute τ:

$$\tau = M(q)\dot{v} + h(q, v)$$

Problem solved by controllers.

Modeling Robots in Contact

Adding Contact Forces

If robot in contact with surrounding $\rightarrow$ contact forces $f \in \mathbb{R}^{n_f}$:

$$M(q)\dot{v} + h(q, v) = \tau + J(q)^\top f,$$

where $J(q) \in \mathbb{R}^{n_f \times n_v} \triangleq$ contact Jacobian:

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where $J(q) \in \mathbb{R}^{n_f \times n_v} \triangleq$ contact Jacobian:

$$J(q) = \frac{\partial c(q)}{\partial q},$$

where $c(q) : \mathbb{R}^{n_q} \rightarrow \mathbb{R}^{n_f} \triangleq$ forward geometry of contact points (i.e. function mapping joint angles to contact point positions).

Robots in Rigid Contact

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Introduce constraints in dynamics:

$$\begin{bmatrix} M & -J^\top & -I \\ J & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ f \\ \tau \end{bmatrix} = \begin{bmatrix} -h \\ -\dot{J}v \end{bmatrix} \tag{1}$$

Inverse VS Forward Dynamics with Rigid Contacts

Forward Dynamics (with constraints)
Given q, v, τ compute $\dot{v}$ and f:

$$\begin{bmatrix} \dot{v} \\ f \end{bmatrix} = \begin{bmatrix} M & -J^\top \\ J & 0 \end{bmatrix}^{-1} \begin{bmatrix} \tau - h \\ -\dot{J}v \end{bmatrix}$$

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Inverse Dynamics (with constraints)
Given $q, v, \dot{v}$ compute τ and f:

$$\begin{bmatrix} \tau \\ f \end{bmatrix} = \begin{bmatrix} I & J^\top \end{bmatrix}^\dagger (M\dot{v} + h),$$

where $\dagger$ represents pseudo-inverse.

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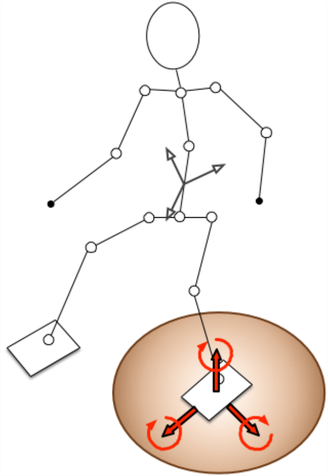
Primitive version of inverse-dynamics control with rigid contacts.

Modeling Legged Robots

Modeling Legged (Floating-Base) Robots



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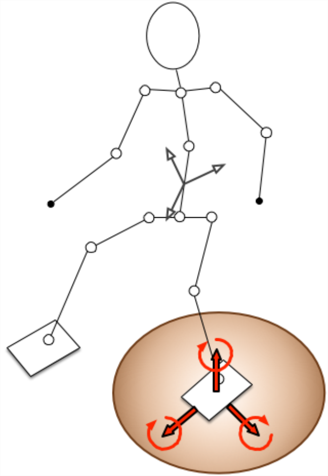

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Add pose (position + orientation) of one link (called base) w.r.t. inertial frame:

$$q = (\underbrace{x_b}_{\text{Base pose}}, \underbrace{q_j}_{\text{Joint angles}})$$

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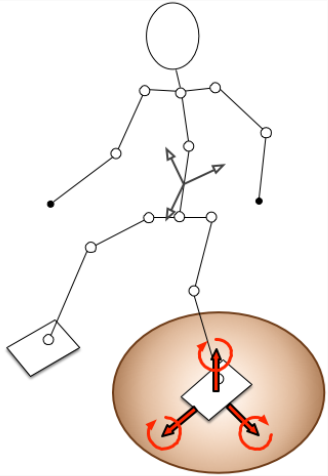

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$$q = (\underbrace{x_b}_{\text{Base pose}}, \underbrace{q_j}_{\text{Joint angles}})$$

Now q sufficient to describe robot configuration in space.

Base Pose

$x_b \in \mathrm{SE}(3) \triangleq$ special Euclidian group, comprising any combination of

- translations: elements of $\mathbb{R}^3$,
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Can represent SO(3) elements with:

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- redundant representations: $\geq$4 elements but free from singularities (e.g., quaternions, rotation matrices)

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We represent SE(3) elements as 7d vectors: 3d for position, 4d for orientation (quaternion).

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Compared to rotation matrices: more compact, numerically stable, and efficient.

Any 3d rotation equivalent to single rotation by angle θ about fixed axis (unit vector $u = (u_x, u_y, u_z)$).

$$\text{quaternion} = (u_x s, u_y s, u_z s, c)$$

where $c = \cos\frac{\theta}{2}$ and $s = \sin\frac{\theta}{2}$. Note that $||\text{quaternion}|| = 1 \quad \forall\, \theta, u$.

Base Velocity

Robot configuration is $q = (x_b, q_j)$, where $x_b = (p_b, o_b) \in \mathbb{R}^7$.

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Angular velocity $\omega_b \in \mathbb{R}^3$ related to time derivative of associated rotation matrix $R_b \in \mathbb{R}^{3\times 3}$ by:

$$\dot{R}_b = \hat{\omega}_b R_b \quad \rightarrow \quad R_b(t) = e^{\hat{\omega}_b t} R_b(0)$$

where $\hat{\omega}_b \in \mathbb{R}^{3\times 3}$ is skew-symmetric matrix associated to ω_b.

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So q and v have different sizes ($n_q = n_v + 1$)

Underactuated Systems

Underactuated systems: less actuators than DoFs:

$$\underbrace{n_{va}}_{\text{number of actuators}} < \underbrace{n_v}_{\text{number of DoFs}}$$

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Assume ordered elements of $q \triangleq (q_u, q_a)$:

- $q_u \in \mathbb{R}^{n_{qu}}$: passive (unactuated) joints,
- $q_a \in \mathbb{R}^{n_{qa}}$: actuated joints.

Similarly, $v \triangleq (v_u, v_a)$, $v_u \in \mathbb{R}^{n_{vu}}$, $v_a \in \mathbb{R}^{n_{va}}$.

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$S \triangleq \begin{bmatrix} 0_{n_{va} \times n_{vu}} & I_{n_{va}} \end{bmatrix}$ is selection matrix:

$$v_a = Sv$$

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For legged robots typically $q_u = x_b$ (all joints are actuated).

Under-Actuated Dynamic

Dynamics of under-actuated mechanical system:

$$M(q)\dot{v} + h(q, v) = S^\top \tau + J(q)^\top f$$

Contrary to fully-actuated case: $\tau \in \mathbb{R}^{n_{va}}$.

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Often decomposed into unactuated and actuated parts:

$$\begin{aligned} M_u(q)\dot{v} + h_u(q, v) &= J_u(q)^\top f \\ M_a(q)\dot{v} + h_a(q, v) &= \tau + J_a(q)^\top f \end{aligned} \tag{2}$$

where

$$M = \begin{bmatrix} M_u \\ M_a \end{bmatrix} \quad h = \begin{bmatrix} h_u \\ h_a \end{bmatrix} \quad J = \begin{bmatrix} J_u & J_a \end{bmatrix} \tag{3}$$

Recap

Manipulator:

$$M(q)\dot{v} + h(q,v) = \tau$$

Manipulator in contact:

$$M(q)\dot{v} + h(q,v) = \tau + J(q)^\top f$$

Legged robot (in contact):

$$M(q)\dot{v} + h(q,v) = S^\top \tau + J(q)^\top f$$

If contacts are rigid:

$$J\dot{v} = -\dot{J}v$$