Joint-Space Control

Optimization-based Control of Legged Robots

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Joint-Space Inverse Dynamics Control

Robot Manipulator

Given (nonlinear) manipulator dynamics:

$$M(q)\dot{v} + h(q, v) = \tau \tag{1}$$

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$$\dot{v}^d = \dot{v}^r - K_d(v - v^r) - K_p(q - q^r) \tag{2}$$

where K_p, K_d are diagonal positive-definite gain matrices.

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A is Hurwitz if K_p and K_d are diagonal and positive-definite $\rightarrow$ $\lim_{t\rightarrow\infty} x(t) = 0 \rightarrow \lim_{t\rightarrow\infty} q(t) = q^r(t)$

Many names for the same approach

This control law:

$$\tau = M(\dot{v}^r - K_d \dot{e} - K_p e) + h \tag{3}$$

is known as:

- Inverse-Dynamics (ID) Control: because based on inverse dynamics computation.
- Computed Torque: because it computes torques needed to get desired accelerations.
- Feedback Linearization (from control theory): because it uses state feedback to linearize closed-loop dynamics.

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Another variant (with similar properties) exists:

$$\tau = M\dot{v}^r - K_d \dot{e} - K_p e + h \tag{4}$$

Other Control Laws for Manipulators

Simpler control laws often used for manipulators.

A common option is PD+gravity compensation:

$$\tau = \underbrace{-K_d \dot{e} - K_p e}_{PD} + \underbrace{g(q)}_{\text{gravity compensation}} \tag{5}$$

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In theory “ID control“ outperforms “PD+gravity”, which outperforms “PID”.

In practice the opposite could occur because of model errors.

Inverse Dynamics Control as Optimization Problem

Inverse Dynamics (ID) Control as Least-Squares Problem

Solution of optimization problem:

$$\begin{aligned}(\tau^*, \dot{v}^*) = \quad & \underset{\tau, \dot{v}}{\text{argmin}} \quad && ||\dot{v} - \dot{v}^d||^2 \\ & \text{subject to} && M\dot{v} + h = \tau \end{aligned} \tag{7}$$

with $\dot{v}^d = \dot{v}^r - K_d \dot{e} - K_p e$

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Problem (7) is Least-Squares Program/Problem (LSP).

Taxonomy of Convex Optimization Problems

Least-Squares Programs (LSP) have:

- linear equality/inequality constraints ($Ax \leq b$, or $Ax = b$)
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LSPs and convex QPs can be solved extremely fast with off-the-shelf softwares
→ We can solve LSP/QPs inside 1 kHz control loops!

Adding Torque Limits to ID Control

Take the ID control LSP:

$$\begin{aligned} \underset{\tau,\dot{v}}{\text{minimize}} \quad & ||\dot{v} - \dot{v}^d||^2 \\ \text{subject to} \quad & M\dot{v} + h = \tau \end{aligned} \tag{9}$$

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Main advantage of optimization: inequality constraints.

Adding Current Limits for Electric Motors

In electric motors current i is proportional to torque τ:

$$i = k_\tau \tau \tag{11}$$

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Adding Joint Velocity Limits

Assuming constant accelerations $\dot{v}$ during time step Δt:

$$v(t + \Delta t) = v(t) + \Delta t\, \dot{v} \tag{13}$$

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Adding Joint Position Limits

Could use same trick for position limits:

$$q(t + \Delta t) = q(t) + \Delta t\, v(t) + \frac{1}{2}\, \Delta t^2\, \dot{v} \tag{15}$$

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Better approaches exist [1, 3, 2], but we don't discuss them here.

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