Task-Space Inverse Dynamics

Quadratic-Programming based Control for Legged Robots

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Very active research topic between 2004 and 2015 [11, 7, 8, 10, 4, 3].
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Very active research topic between 2004 and 2015 [11, 7, 8, 10, 4, 3]. Now not so active anymore (i.e. problem solved), but widely used.
Schedule

1. Theory (≈ 1/1.5 hours)
2. Implementation (≈ 0.5/1 hour)
3. Coding (≈ 0.5/1 hour)
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Options for coding:

- install TSID in Nicolas’s VM (recommended)
  - go to https://github.com/stack-of-tasks/tsid/issues
  - open issue 28 (should be the latest)
  - execute list of commands I posted
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- use my 11 GB VM (prepared with VMware Fusion)
- install TSID and dependencies on your machine
  - TSID branch master → Pinocchio branch master (same as robotpkg binaries)
  - TSID branch pinocchio-v2 → Pinocchio branch devel
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A system is fully actuated if the number of actuators is equal to the number of degrees of freedom (e.g., manipulator).

A system is under actuated if the number of actuators is less than the number of degrees of freedom (e.g., legged robot, quadrotor).
1. Actuation Models

2. Task Models

3. Optimization-Based Control

4. Multi-Task Control

5. Computational Aspects
Actuation Models
Everything starts with a model (not really, but here it does).
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Everything starts with a model (not really, but here it does). Appropriate model choice depends on both robot and task. Let us discuss three models for the robot actuators:

- velocity source
- acceleration source
- torque source
Assume motors are velocity sources.

- Good approximation for hydraulic actuators.
- Good approximation for electric motors only in certain conditions (e.g., industrial manipulators, not for legged robots).
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Robot state $x$ is described by its configuration $q$.

Control inputs $u$ are robot velocities $v_q$.

Dynamic for fully-actuated systems is a simple integrator:

$$v_q = u$$
Assume motors are acceleration sources.

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- Good approximation for electric motors as long as large contact forces are not involved.

Robot state $x$ is described by its configuration $q$ and its velocity $v_q$:

$$x \triangleq (q, v_q)$$

Control inputs $u$ are robot accelerations $\dot{v}_q$.

Dynamic for fully-actuated systems is a double integrator:

$$\begin{bmatrix} v_q \\ \dot{v}_q \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ v_q \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$
Assume motors are torque sources. Good approximation for electric motors.

Robot state $x$ is described by its configuration $q$ and its velocity $v_q$:

$$x \triangleq (q, v_q)$$

Control inputs $u$ are motor torques $\tau$. 
The dynamic equation of a fully-actuated mechanical system is:

\[ M(q) \dot{v}_q + h(q, v_q) = \tau + J(q)^\top f, \]

where \( M(q) \in \mathbb{R}^{n_v \times n_v} \) is the mass matrix, \( h(q, v_q) \in \mathbb{R}^{n_v} \) are the bias forces, \( \tau \in \mathbb{R}^{n_v} \) are the joint torques, \( f \in \mathbb{R}^{n_f} \) are the contact forces, and \( J(q) \in \mathbb{R}^{n_f \times n_v} \) is the contact Jacobian.
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**Bias forces** are sometimes decomposed in two components:

\[ h(q, v_q) = C(q, v_q)v_q + g(q) \]

- \( C(q, v_q)v_q \) contains Coriolis and centrifugal effects
- \( g(q) \) contains the gravity forces
Underactuated systems (such as legged robots) have less actuators than degrees of freedom (DoFs). Calling $n_{va}$ the number of actuators, and $n_v$ the number of DoFs, we have $n_{va} < n_v$. 
Torque Control: Under-Actuated Systems

Underactuated systems (such as legged robots) have less actuators than degrees of freedom (DoFs). Calling \( n_{va} \) the number of actuators, and \( n_v \) the number of DoFs, we have \( n_{va} < n_v \).

Assume elements of \( q \) are ordered, \( q \triangleq (q_u, q_a) \), where:

- \( q_u \in \mathbb{R}^{n_{qu}} \) are the passive (unactuated) joints,
- \( q_a \in \mathbb{R}^{n_{qa}} \) are the actuated joints.

Similarly, \( \nu_q \triangleq (\nu_u, \nu_a) \), where \( \nu_u \in \mathbb{R}^{n_{vu}} \) and \( \nu_a \in \mathbb{R}^{n_{va}} \).
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$S \triangleq \begin{bmatrix} 0_{n_{va} \times n_{vu}} & I_{n_{va}} \end{bmatrix}$ is a selection matrix associated to the actuated joints:

$$\nu_a = Sv_q$$
The dynamic of an under-actuated mechanical system is:

\[ M(q)\dot{q} + h(q, \nu_q) = S^\top \tau + J(q)^\top f, \]

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where, contrary to the fully-actuated case, \( \tau \in \mathbb{R}^{n_{va}} \).

This dynamic is often decomposed into unactuated and actuated parts:

\[
\begin{align*}
M_u(q) \dot{\nu}_q + h_u(q, \nu_q) &= J_u(q)^\top f \\
M_a(q) \dot{\nu}_q + h_a(q, \nu_q) &= \tau + J_a(q)^\top f
\end{align*}
\]

where

\[
M = \begin{bmatrix} M_u \\ M_a \end{bmatrix} \quad h = \begin{bmatrix} h_u \\ h_a \end{bmatrix} \quad J = \begin{bmatrix} J_u & J_a \end{bmatrix}
\]
Task Models
IDEA: Describe task to be performed (i.e. control objective) as a function to minimize (similar to optimal control).

**Error Function**

\[ e(x, u, t) = |y(x, u) - y^*(t)| \]

Note: Contrary to an optimal control cost function, \( e \) does not depend on the state-control trajectory, but only on the instantaneous state-control value.
IDEA: Describe task to be performed (i.e. control objective) as a function to minimize (similar to optimal control).

Without loss of generality, assume this function measures an error between real and reference value of output $y \in \mathbb{R}^m$:

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N.B.
Contrary to an optimal control cost function, $e$ does not depend on the state-control trajectory, but only on the instantaneous state-control value.
Consider three kinds of task functions:

- Affine functions of control inputs: \( e(u, t) = A_u u - a(t) \)
- Nonlinear functions of robot velocities: \( e(v_q, t) = y(v_q) - y^*(t) \)
- Nonlinear functions of robot configuration: \( e(q, t) = y(q) - y^*(t) \)
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**Issue**

Control inputs \( u \) can be instantaneously changed, but that is not the case for the state \( x \).
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### Issue
Control inputs \( u \) can be instantaneously changed, but that is not the case for the state \( x \).

### Solution
Impose dynamic of task function \( e(x, t) \) such that \( \lim_{t \to \infty} e(x, t) = 0 \)
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Let us impose a \textbf{first-order} linear dynamic:

\[
\dot{e} = -Ke
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\[
\frac{\partial y}{\partial v_q} \dot{v}_q - \dot{y}^* = -Ke
\]

\textit{Jacobian}
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\]

\[
J \dot{v}_q = \dot{y}^* - Ke
\]

(3)

We got an affine function of the accelerations \( \dot{v}_q \).

N.B. We could also impose a nonlinear dynamic, but in practice a linear dynamic is ok for most cases.
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Let us impose a second-order linear dynamic:

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\]
Consider a task function: \( e(q, t) = y(q) - y^*(t) \).

Let us impose a second-order linear dynamic:

\[
\ddot{e} = -K_e - D\dot{e} \\
J\dot{v}_q + Jv_q - \ddot{y}^* = -K_e - D\dot{e} \\
J \dot{v}_q = \ddot{y}^* - Jv_q - Ke - D\dot{e} \\
\begin{aligned}
A_v \dot{v}_q &= \ddot{y}^* - Jv_q - Ke - D\dot{e}
\end{aligned}
\]

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J\dot{v}_q + \dot{J}v_q - \ddot{y}^* = -Ke - D\dot{e}
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\[
Jv_q = \ddot{y}^* - \dot{J}v_q - Ke - D\dot{e}
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We could also impose a nonlinear dynamic, but in practice a linear dynamic is ok for most cases.
Task functions can depend either on $u$, or on $x \triangleq (q, v_q)$.

$$g(y) \triangleq \begin{bmatrix} A v & A u \end{bmatrix} \begin{bmatrix} \dot{v}_q & u \end{bmatrix} y - a$$
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Functions of \( x \) can be **nonlinear**, but cannot be directly imposed.

- For functions of \( v_q \) we can impose first derivative.
- For functions of \( q \) we can impose second derivative.

In any case, we end up with an **affine** function of \( \dot{v}_q \) and \( u \):

\[
g(y) \triangleq \begin{bmatrix} A_v & A_u \end{bmatrix} \begin{bmatrix} \dot{v}_q \\ u \end{bmatrix} - a
\]

\[A\]
Optimization-Based Control
IDEA: formulate control problem as an optimization problem (similar to optimal control).
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Key elements are:

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**IDEA:** formulate control problem as an optimization problem (similar to optimal control).

Key elements are:

- **state:** \( x \triangleq (q, v_q) \)
- **control:** \( u \triangleq \tau \)
- **dynamic (no contacts):** \( M \dot{v}_q + h = S^\top \tau \)
- **task function to minimize:** \( ||g(y)||^2 \triangleq ||Ay - a||^2 \)
Formulate optimization problem to find control inputs that minimize task function:

$$\text{minimize} \quad \| Ay - a \|^2$$

subject to

$$\begin{bmatrix} M & -S^T \end{bmatrix} y = -h$$

(5)
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\]

Equality constraints are affine and cost function is convex quadratic → Problem is a **Quadratic Program (QP)**.
Formulate optimization problem to find control inputs that minimize task function:

\[
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\]

subject to

\[
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M & -S^T
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Equality constraints are affine and cost function is convex quadratic

\[\rightarrow\] Problem is a Quadratic Program (QP).

\textbf{N.B.}

To be precise, cost function is 2-norm of affine function, which is a special kind of convex quadratic function (linear term \(A^T a\) is in range space of Hessian \(A^T A\)) \[\rightarrow\] Problem is a Least-Squares Problem (LSP).
If system is in contact with environment, its dynamic must account for contact forces $f$. 
If system is in **contact** with environment, its dynamic must account for contact forces $f$.

If contacts are **soft**, measured/estimated contact forces $\hat{f}$ can be easily included:

$$\text{minimize } y = (\dot{\mathbf{v}_q}, \tau) \quad ||Ay - a||^2$$

subject to

$$\begin{bmatrix} M & -S^\top \end{bmatrix} y = -h + J^\top \hat{f}$$

(6)
If contacts are rigid, they constrain the motion.
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\[ c(q) = 0 \quad \iff \quad \text{Contact points do not move} \]
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To express the constraints as functions of the problem variables we must differentiate them twice:

$$Jv_q = 0 \iff \text{Contact point velocities are null}$$
$$J\dot{v}_q + Jv_q = 0 \iff \text{Contact point accelerations are null}$$
TSID for Robots in Rigid Contact

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Introduce contact forces and contact constraints in optimization problem:

\[
\begin{align*}
\text{minimize } &\|Ay - a\|^2 \\
\text{subject to } &\begin{bmatrix} J & 0 & 0 \\ M & -J^\top & -S^\top \end{bmatrix} y = \begin{bmatrix} -Jv_q \\ -h \end{bmatrix}
\end{align*}
\]

(7)
So far we have seen only Equality-Constrained LSP (ECLSP).

Unconstrained LSP can be solved using pseudo-inverses, for instance:

\[ y^* = \arg\min_y ||Ay - a||^2 \iff y^* = A^\dagger a \]

Also ECLSP can be solved using pseudo-inverses, for instance:

\[ y^* = \arg\min_y ||Ay - a||^2 \iff y^* = B^\dagger b + N_B (A^\dagger N_B A^\dagger) (a - AB^\dagger b) \]

subject to \( By = b \)

where \( N_B = I - B^\dagger B \) is the null-space projector of \( B \).

QUESTION: if we can solve ECLSP with pseudo-inverses, why should we use a QP solver?
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Main benefit of QP solvers (over pseudo-inverses) is that they can handle inequality constraints.
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We can account for any inequality affine in problem variables $y$, such as:

- joint torque bounds: $\tau^\text{min} \leq \tau \leq \tau^\text{max}$
- (linearized) force friction cones: $Bf \leq 0$
- joint position-velocity bounds (after nontrivial transformation into acceleration bounds [1]): $\dot{\nu}_q^\text{min} \leq \dot{\nu}_q \leq \dot{\nu}_q^\text{max}$
Multi-Task Control
Complex robots are typically redundant with respect to the main task they must perform.
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Redundancy can be used to execute secondary tasks, but how to incorporate them in the optimization problem?
Weighted Multi-Objective Optimization

Assume robot must perform $N$ tasks, each defined by a task function

$$g_i(y) = \|A_i y - a_i\|^2 \quad i = 1 \ldots N$$

Simplest strategy: sum all functions using user-defined weights $w_i$:

$$\min_{y} (\dot{v} q, f, \tau)$$

subject to

$$\begin{bmatrix} J_0 & 0 & \mathbf{0} \\ \mathbf{0} & M & -J^T \\ \mathbf{0} & \mathbf{0} & S^T \end{bmatrix} y = \begin{bmatrix} -\dot{J} v q - h \end{bmatrix}$$

**PROS** Problem remains standard computationally-efficient LSP.

**CONS** Finding proper weights can be hard, too large/small weights can lead to numerical issues.
Weighted Multi-Objective Optimization

Assume robot must perform $N$ tasks, each defined by a task function

$$g_i(y) = \|A_i y - a_i\|^2 \quad i = 1 \ldots N$$

Simplest strategy: sum all functions using user-defined weights $w_i$:

$$\minimize_{y=(\dot{v}_q,f,\tau)} \sum_{i=1}^{N} w_i g_i(y)$$

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Alternative strategy: order task functions according to priority, that is

- task 1 is infinitely more important than task 2
- ...
- task N-1 is infinitely more important than task N

Solve a sequence (cascade) of N optimization problems, from $i = 1$:

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Computational Aspects
TSID needs to solve a QP at each control loop (embedded optimization, same spirit as MPC).

For $n_v$ DoFs, $n_{va}$ motors, and $n_f$ contact constraints the QP has:

- $n_v + n_{va} + n_f$ variables ($\approx 70$ for humanoid)
- $n_v + n_f$ equality constraints ($\approx 40$ for humanoid)
- $n_v + n_{va} + 4n_f$ inequality constraints (assuming friction cones are approximated with 4-sided pyramids)

Computational cost dominated by Hessian (Cholesky) decomposition: $O(n^3)$, with $n$ the number of variables.

Questions:
- Can we solve such a problem in 1 ms?
- Is there a way to speed up computation?
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IDEA: Exploit structure of problem to make computation faster.
Reformulating Optimization Problem

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Equality constraints have special structure:

\[
\begin{bmatrix}
J & 0 & 0 \\
M_u & -J_u^T & -0 \\
M_a & -J_a^T & -1
\end{bmatrix}
\begin{bmatrix}
\dot{v}_q \\
\tau
\end{bmatrix}
= 
\begin{bmatrix}
-J\dot{v}_q \\
-f \\
-h_u \\
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  f \\
  \tau
\end{bmatrix}
= 
\begin{bmatrix}
  -\dot{J}v_q \\
  -h_u \\
  -h_a
\end{bmatrix}
\]

Identity matrix is easy to invert → We can easily express \( \tau \) as affine function of other variables.

\[
\begin{bmatrix}
  \dot{v}_q \\
  f \\
  \tau
\end{bmatrix}
= 
\begin{bmatrix}
  I & 0 & 0 \\
  0 & I & 0 \\
  M_a & -J_a^T & h_a
\end{bmatrix}
\begin{bmatrix}
  \dot{v}_q \\
  f \\
  \tau
\end{bmatrix}
+ 
\begin{bmatrix}
  0 \\
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Original problem:

minimize \( y \) \( | |Ay - a| |^2 \)

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Use \( y = D\bar{y} + d \) to reformulate problem [5]:

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We have removed \( n_{va} \) variables and \( n_{va} \) equality constraints.
Can we improve even more?

In theory, yes:
- for floating-base robots, remove first 6 variables of \( \dot{v} \)
  exploiting structure of first 6 columns of \( M \)
- remove (either all \([7, 9]\) or some \([2]\)) force variables by projecting dynamics in null space of \( J \)

BUT these tricks either limit the expressiveness of the problem, or lead to small improvements (while making the software more complex).

My opinion: probably not worth it!
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So far we have assumed output function $y(x, u) \in \mathbb{R}^m$. What if instead $y(x, u) \in SE(3)$? (very common in practice for $y(q)$)

SOLUTION Represent SE(3) elements using homogeneous matrices $y \in \mathbb{R}^{4 \times 4}$ and redefine error function:

$$e(q, t) = \log(y^*(t) - 1 y(q)),$$

where log is the pseudo-inverse operation of the matrix exponential (i.e. exponential map): it transforms a displacement into a twist.
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**SOLUTION** Represent $SE(3)$ elements using homogeneous matrices $y \in \mathbb{R}^{4 \times 4}$ and redefine error function:

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A. Del Prete. 
Joint Position and Velocity Bounds in Discrete-Time Acceleration / Torque Control of Robot Manipulators. 

Partial Force Control of Constrained Floating-Base Robots. 

Prioritized Motion-Force Control of Constrained Fully-Actuated Robots: ”Task Space Inverse Dynamics”. 
Hierarchical Quadratic Programming: Fast Online Humanoid-Robot Motion Generation.  

Momentum control with hierarchical inverse dynamics on a torque-controlled humanoid.  

O. Khatib.  
A unified approach for motion and force control of robot manipulators: The operational space formulation.  


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This has been shown not to be the case, but not everybody is aware of/agrees with this, so...beware!